

# SCHRÖDINGER EQUATION ON LOCALLY SYMMETRIC SPACES

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*Dedicated to the memory of Georges Georganopoulos*

**ABSTRACT.** We prove dispersive and Strichartz estimates for Schrödinger equations on a class of locally symmetric spaces  $\Gamma \backslash X$ , where  $X = G/K$  is a symmetric space and  $\Gamma$  is a torsion free discrete subgroup of  $G$ . We deal with the cases when either  $X$  has rank one or  $G$  is complex. We present Strichartz estimates applications to the well-posedness and scattering for nonlinear Schrödinger equations.

## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let  $M$  be a Riemannian manifold and denote by  $\Delta$  its Laplace-Beltrami operator. The nonlinear Schrödinger equation (NLS) on  $M$

$$(1) \quad \begin{cases} i\partial_t u(t, x) + \Delta_x u(t, x) = F(u(t, x)), \\ u(0, x) = f(x), \end{cases}$$

has been extensively studied the last thirty years. Its study relies on precise estimates of the kernel  $s_t$  of the Schrödinger operator  $e^{it\Delta}$ , the heat kernel of pure imaginary time. The estimates of  $s_t$  allow us to obtain *dispersive estimates* of the operator  $e^{it\Delta}$  of the form

$$(2) \quad \|e^{it\Delta}\|_{L^{\tilde{q}'}(M) \rightarrow L^q(M)} \leq c\psi(t), \quad t \in \mathbb{R},$$

for all  $q, \tilde{q} \in (2, \infty]$ , where  $\psi$  is a positive function and  $\tilde{q}'$  is the conjugate of  $\tilde{q}$ .

Dispersive estimates of  $e^{it\Delta}$  as above, allow us to obtain *Strichartz estimates* of the solutions  $u(t, x)$  of (1):

$$(3) \quad \|u\|_{L^p(\mathbb{R}; L^q(M))} \leq c \left\{ \|f\|_{L^2(M)} + \|F\|_{L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(M))} \right\},$$

for all pairs  $\left(\frac{1}{p}, \frac{1}{q}\right)$  and  $\left(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}}\right)$  which lie in a certain triangle.

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Strichartz estimates have applications to well-posedness and scattering theory for the NLS equation.

In the case of  $\mathbb{R}^n$ , the first such estimate was obtained by Strichartz himself [31] in a special case. Then, Ginibre and Velo [14] obtained the complete range of estimates except the case of endpoints which were proved by Keel and Tao [24].

In view of the important applications to nonlinear problems, many attempts have been made to study the dispersive properties for the corresponding equations on various Riemannian manifolds (see e.g., [5, 6, 8, 9, 10, 11, 17, 22, 29, 30] and references within). More precisely, dispersive and Strichartz estimates for the Schrödinger equation on real hyperbolic spaces have been stated by Banica [8], Pierfelice [29, 30], Banica et al. [9], Anker and Pierfelice [5], Ionescu and Staffilani [22]. In a recent paper Anker, Pierfelice and Vallarino [6] treat NLS in the context of Damek-Ricci spaces, which include all rank one symmetric spaces of noncompact type.

In the present work we treat NLS equations on a class of locally symmetric spaces.

**1.1. The class  $(S)$  of locally symmetric spaces.** In this section we describe the class of locally symmetric spaces on which we shall treat NLS equations.

For the statement of the results we need to introduce some notation. For more details see Section 2. Let  $G$  be a semisimple Lie group, connected, noncompact, with finite center and  $K$  be a maximal compact subgroup of  $G$ . We denote by  $X$  the Riemannian symmetric space  $G/K$ .

Denote by  $\mathfrak{g}$  and  $\mathfrak{k}$  the Lie algebras of  $G$  and  $K$ . Let also  $\mathfrak{p}$  be the subspace of  $\mathfrak{g}$  which is orthogonal to  $\mathfrak{k}$  with respect to the Killing form. The Killing form induces a  $K$ -invariant scalar product on  $\mathfrak{p}$  and hence a  $G$ -invariant metric on  $G/K$ . Denote by  $\Delta$  the Laplace-Beltrami operator on  $X$  and by  $d(.,.)$  the Riemannian distance and by  $dx$  the associated measure on  $X$ .

Let  $\Gamma$  be a discrete torsion free subgroup of  $G$ . Then the locally symmetric space  $M = \Gamma \backslash X$ , equipped with the projection of the canonical Riemannian structure of  $X$ , becomes a Riemannian manifold. We denote also by  $\Delta$  the laplacian on  $M$ , by  $d(.,.)$  the Riemannian distance and by  $dx$  the associated measure on  $M$ . It is important to note that in general, locally symmetric spaces have not bounded geometry since the injectivity radius of  $M$  is not in general strictly positive, [13].

Fix  $\mathfrak{a}$  a maximal abelian subspace of  $\mathfrak{p}$  and denote by  $\mathfrak{a}^*$  the real dual of  $\mathfrak{a}$ . If  $\dim \mathfrak{a} = 1$ , we say that  $X$  has *rank one*. Let  $\Sigma \subset \mathfrak{a}^*$ , be

the root system of  $(\mathfrak{g}, \mathfrak{a})$ . Denote by  $W$  the Weyl group associated to  $\Sigma$  and choose a set  $\Sigma^+$  of positives roots. Denote by  $\rho$  the half sum of positive roots counted with their multiplicities. Let  $\mathfrak{a}^+ \subset \mathfrak{a}$  be the corresponding positive Weyl chamber and let  $\overline{\mathfrak{a}^+}$  be its closure. Set

$$\rho_m = \min_{H \in \overline{\mathfrak{a}^+}, |H|=1} \rho(H).$$

Note that in the rank one case  $\rho_m = |\rho|$ .

Denote by  $s_t$  the fundamental solution of the Schrödinger equation on the symmetric space  $X$ :

$$i\partial_t s_t(x, y) = \Delta s_t(x, y), \quad t \in \mathbb{R}, \quad x, y \in X.$$

Then  $s_t$  is a  $K$ -bi-invariant function and the Schrödinger operator  $S_t = e^{it\Delta}$  on  $X$  is defined as a convolution operator:

$$(4) \quad S_t f(x) = \int_G f(y) s_t(y^{-1}x) dy = (f * s_t)(x), \quad f \in C_0^\infty(X).$$

Using that  $s_t$  is  $K$ -bi-invariant, we deduce that if  $f \in C_0^\infty(M)$ , then  $S_t f$  is right  $K$ -invariant and left  $\Gamma$ -invariant i.e. a function on the locally symmetric space  $M$ . Thus the Schrödinger operator  $\widehat{S}_t$  on  $M$  is also defined by formula (4).

The first ingredient for the proof of the dispersive estimate (2) are precise estimates of the kernel  $s_t$ . In the present work we deal with the cases when either  $X$  has rank one or  $G$  is complex. The reason is that in these cases the expression of the spherical Fourier transform allow us to obtain precise estimates of kernel  $s_t$ . They are obtained in [6, Section 3] in the context of rank one symmetric spaces and in Section 4 in the case  $G$  complex.

Set

$$(5) \quad \widehat{s}_t(x, y) = \sum_{\gamma \in \Gamma} s_t(x, \gamma y).$$

Recall that the critical exponent  $\delta(\Gamma)$  is defined by

$$\delta(\Gamma) = \inf \{ \alpha > 0 : P_\alpha(x, y) < \infty \},$$

where

$$P_\alpha(x, y) = \sum_{\gamma \in \Gamma} e^{-\alpha d(x, \gamma y)}$$

are the Poincaré series, [32]. Note that  $\delta(\Gamma) \leq 2|\rho|$ .

In Section 5, we show that the series (5) converges when  $\delta(\Gamma) < \rho_m$  and that  $\widehat{S}_t$  is an integral operator on  $M$  with kernel  $\widehat{s}_t(x, y)$ :

$$(6) \quad \widehat{S}_t f(x) = \int_M f(y) \widehat{s}_t(x, y) dy.$$

The expression (6) as well as the norm estimates of  $\widehat{s}_t$  obtained in the same section are the second ingredient for the proof of the dispersive estimate (2) of the operator  $\widehat{S}_t$ .

The third ingredient is the following analogue of Kunze and Stein phenomenon on locally symmetric spaces, proved in [27], and presented in detail in Section 3. Let  $\lambda_0$  be the bottom of the  $L^2$ -spectrum of  $-\Delta$  on  $M$ . Then in [27] it is proved that there exists a vector  $\eta_\Gamma$  on the Euclidean sphere  $S\left(0, (|\rho|^2 - \lambda_0)^{1/2}\right)$  of  $\mathfrak{a}^*$ , such that for all  $p \in (1, \infty)$  and for every  $K$ -bi-invariant function  $\kappa$ , the convolution operator  $*|\kappa|$  with kernel  $|\kappa|$  satisfies the estimate

$$(7) \quad \|*|\kappa|\|_{L^p(M) \rightarrow L^p(M)} \leq \int_G |\kappa(g)| \varphi_{-i\eta_\Gamma}(g)^{s(p)} dg,$$

where  $\varphi_\lambda$  is the spherical function with index  $\lambda$  and

$$s(p) = 2 \min((1/p), (1/p')).$$

Note that Leuzinger [26] proved that if  $G$  has no compact factors, its center is trivial and  $\delta(\Gamma) \leq \rho_m$ , then  $\lambda_0 = |\rho|^2$ . In this case,  $\eta_\Gamma = 0$  in (7). Note that this is also the case if  $M$  has rank one and  $\delta(\Gamma) \leq \rho_m$ . For simplicity we shall assume that *all locally symmetric spaces* we deal with, satisfy  $\lambda_0 = |\rho|^2$ .

**Definition 1.** *We say that the locally symmetric space  $M = \Gamma \backslash G/K$  belongs in the class  $(S)$  if*

- (i)  $\delta(\Gamma) < \rho_m$ , and
- (ii) *for every  $K$ -bi-invariant function  $\kappa$  the estimate (7) is satisfied with  $\eta_\Gamma = 0$ .*

In particular, if  $M \in (S)$  and either  $M$  has rank one or  $G$  is complex, we say that  $M$  belongs in the class  $(S_0)$ . Note that if  $M \in (S_0)$ , then the three ingredients for the proof of the dispersive estimate (2) of the operator  $\widehat{S}_t$  are available. In the present work we treat the case  $M \in (S_0)$ .

As it is explained in Section 3, the estimate (7) is valid if  $M$  belongs in one of the following three classes:

- (i).  $\Gamma$  is a lattice i.e.  $\text{vol}(\Gamma \backslash G) < \infty$ ,
- (ii).  $G$  possesses Kazhdan's property (T). Recall that  $G$  has property (T) iff  $G$  has no simple factors locally isomorphic to  $SO(n, 1)$  or

$SU(n, 1)$ , [16, ch. 2]. In this case  $\Gamma \backslash G/K \in (S)$  for all discrete subgroups  $\Gamma$  of  $G$  with  $\delta(\Gamma) < |\rho|$ .

Recall also that non-compact rank one symmetric spaces are the real, complex and quaternionic hyperbolic spaces, denoted  $H^n(\mathbb{R})$ ,  $H^n(\mathbb{C})$ ,  $H^n(\mathbb{H})$  respectively, and the octonionic hyperbolic plane  $H^2(\mathbb{O})$ . They have the following representation as quotients:

$$\begin{aligned} H^n(\mathbb{R}) &= SO(n, 1)/SO(n), \quad H^n(\mathbb{C}) = SU(n, 1)/SU(n), \\ H^n(\mathbb{H}) &= Sp(n, 1)/Sp(n), \text{ and } H^2(\mathbb{O}) = F_4^{-20}/Spin(9). \end{aligned}$$

So,  $\Gamma \backslash H^n(\mathbb{H})$  and  $\Gamma \backslash H^2(\mathbb{O})$  belong in the class  $(S)$  for *all* discrete subgroups  $\Gamma$  of  $Sp(n, 1)$  and  $F_4^{-20}$  respectively such that  $\delta(\Gamma) < |\rho|$ .

(iii) all quotients  $\Gamma \backslash H^n(\mathbb{R})$  and  $\Gamma \backslash H^n(\mathbb{C})$  with  $\Gamma$  *amenable* with  $\delta(\Gamma) < |\rho|$ , even if they have infinite volume.

**1.2. Dispersive and Strichartz estimates on locally symmetric spaces.** The main result of the present paper is the following dispersive estimate.

**Theorem 2.** *Assume that  $M \in (S_0)$ . If  $q, \tilde{q} \in (2, \infty]$ , then for  $|t| < 1$ ,*

$$\|\widehat{S}_t\|_{L^{\tilde{q}'}(M) \rightarrow L^q(M)} \leq c|t|^{-n \max\{(1/2)-(1/q), (1/2)-(1/\tilde{q})\}},$$

*while for  $|t| \geq 1$ ,*

$$\|\widehat{S}_t\|_{L^{\tilde{q}'}(M) \rightarrow L^q(M)} \leq c|t|^{-3/2},$$

*if  $M$  has rank one, and*

$$\|\widehat{S}_t\|_{L^{\tilde{q}'}(M) \rightarrow L^q(M)} \leq c|t|^{-n/2},$$

*if  $G$  is complex.*

Consider the following Cauchy problem for the linear inhomogeneous Schrödinger equation on  $M$ :

$$(8) \quad \begin{cases} i\partial_t u(t, x) + \Delta u(t, x) = F(t, x), \\ u(0, x) = f(x). \end{cases}$$

Combining the above dispersive estimate with the classical  $TT^*$  method developed by Kato [23], Ginibre and Velo [14] and Keel and Tao [24], we obtain Strichartz estimates for the solutions  $u(t, x)$  of (8). Consider the triangle

$$(9) \quad T_n = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in (0, \frac{1}{2}] \times (0, \frac{1}{2}] : \frac{2}{p} + \frac{n}{q} \geq \frac{n}{2} \right\} \cup \left\{ (0, \frac{1}{2}) \right\}.$$

**Theorem 3.** *Assume that  $M \in (S_0)$ . If  $(p, q)$  and  $(\tilde{p}, \tilde{q})$  are admissible pairs in the triangle  $T_n$ , then there exists a constant  $c > 0$  such that the following Strichartz estimate holds for the solutions  $u(t, x)$  of the Cauchy problem (8):*

$$(10) \quad \|u\|_{L_t^p L_x^q} \leq c \left\{ \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} \right\}.$$

As it is noticed in [5, 6] the above set  $T_n$  of admissible pairs is much wider than the admissible set in the case of  $\mathbb{R}^n$  which is just the lower edge of the triangle. This phenomenon was already observed for hyperbolic spaces in [9, 22].

The paper is organised as follows. In Section 2 we describe the geometric context of symmetric spaces and we present the spherical Fourier transform. In Section 3 we present an analogue of Kunze and Stein phenomenon for convolution operators on a class of locally symmetric spaces proved in [27] by Lohoué and Marias. In Section 4 we prove pointwise estimates of the Schrödinger kernel on symmetric spaces  $X = G/K$  when  $G$  is complex. In Section 5 we study the Schrödinger operator on  $M$  and we prove norm estimates for its kernel. In Section 6 we prove dispersive estimates for the Schrödinger operator on  $M$  and we give the proofs of Theorems 2 and 3. Finally, we apply Strichartz estimates to study well-posedness and scattering for NLS equations.

## 2. PRELIMINARIES

In this section we shall recall some basic facts about symmetric spaces of noncompact type. For more details see [18, 2, 4, 20, 21, 27].

Let  $A$  be the analytic subgroup of  $G$  with Lie algebra  $\mathfrak{a}$ . Let  $\mathfrak{a}^+ \subset \mathfrak{a}$  be a positive Weyl chamber and let  $\overline{\mathfrak{a}_+}$  be its closure. Put  $A^+ = \exp \mathfrak{a}_+$ . Its closure in  $G$  is  $\overline{A^+} = \exp \overline{\mathfrak{a}_+}$ . We have the Cartan decomposition

$$G = K(\overline{A^+})K = K(\exp \overline{\mathfrak{a}_+})K.$$

Let  $k_1$ ,  $k_2$  and  $\exp H$  be the components of  $g \in G$  in  $K$  and  $\exp \mathfrak{a}$  according to the Cartan decomposition. Then  $g$  is written as  $g = k_1(\exp H)k_2$ . According to Cartan decomposition, the Haar measure on  $G$  is written as

$$(11) \quad \int_G f(g) dg = c \int_K dk_1 \int_{\mathfrak{a}^+} \delta(H) dH \int_K f(k_1(\exp H)k_2) dk_2,$$

where

$$\delta(H) = \prod_{\alpha \in \Sigma^+} \sinh^{m_\alpha} \alpha(H),$$

with  $m_\alpha = \dim \mathfrak{g}_\alpha$ , and

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\},$$

is the root space associated to the root  $\alpha \in \Sigma^+$ . Note that

$$(12) \quad \delta(H) \leq ce^{2\rho(H)}, \quad H \in \mathfrak{a}_+.$$

If  $G$  has real rank one, then it is well known [20] that the root system  $\Sigma$  is either of the form  $\{-\alpha, \alpha\}$  or of the form  $\{-\alpha, -2\alpha, \alpha, 2\alpha\}$ . Thus  $\rho = \alpha/2$  or  $\rho = (3/2)\alpha$ . Thus  $\rho_m = \min_{H \in \overline{\mathfrak{a}_+}, |H|=1} \rho(H) = \rho$ .

Let  $H_0$  be the unique element of  $\mathfrak{a}$  with the property that  $\alpha(H_0) = 1$ . Set  $a(s) = \exp(sH_0)$ ,  $s \in \mathbb{R}$ . Then  $a : \mathbb{R} \rightarrow A$  is a diffeomorphism and we identify  $A = \exp \mathfrak{a}$  with  $\mathbb{R}$  via  $a$ . We also normalize the Killing form on  $\mathfrak{g}$  such that

$$d(a(s) \cdot \mathbf{0}, \mathbf{0}) = |s|, \quad \text{for all } s \in \mathbb{R},$$

where  $\mathbf{0} = \{K\}$  is the origin of  $X$ .

In this case we have that  $A_+ = \{a(s) : s \geq 0\} \simeq \mathbb{R}^+$  and any  $K$ -bi-invariant function  $\kappa$  is identified with the function  $\kappa : \mathbb{R}_+ \rightarrow \mathbb{C}$ , given by  $\kappa(s) = \kappa(a(s) \cdot \mathbf{0})$ . So, if  $\kappa$  is  $K$ -bi-invariant, then from (11) one has

$$(13) \quad \int_G \kappa(g) dg \leq c \int_{\mathbb{R}^+} \kappa(s) e^{2|\rho|s} ds.$$

**2.1. The spherical Fourier transform.** Denote by  $S(K \backslash G / K)$  the Schwartz space of  $K$ -bi-invariant functions on  $G$ . The spherical Fourier transform  $\mathcal{H}$  is defined by

$$\mathcal{H}f(\lambda) = \int_G f(x) \varphi_\lambda(x) dx, \quad \lambda \in \mathfrak{a}^*, \quad f \in S(K \backslash G / K),$$

where  $\varphi_\lambda(x)$  are the elementary spherical functions on  $G$ , which by a theorem of Harish-Chandra, [18, p.418], are given by

$$(14) \quad \varphi_\lambda(x) = \int_K e^{(\pm i\lambda + \rho)(H(x^\pm k))} dk, \quad x \in G, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*.$$

Let  $S(\mathfrak{a}^*)$  be the usual Schwartz space on  $\mathfrak{a}^*$ , and let us denote by  $S(\mathfrak{a}^*)^W$  the subspace of  $W$ -invariants in  $S(\mathfrak{a}^*)$ . Then, by a celebrated theorem of Harish-Chandra,  $\mathcal{H}$  is an isomorphism between  $S(K \backslash G / K)$  and  $S(\mathfrak{a}^*)^W$  and its inverse is given by

$$(15) \quad (\mathcal{H}^{-1}f)(x) = c \int_{\mathfrak{a}^*} f(\lambda) \varphi_{-\lambda}(x) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}, \quad x \in G, \quad f \in S(\mathfrak{a}^*)^W,$$

where  $\mathbf{c}(\lambda)$  is the Harish-Chandra function.

Recall that the Schrödinger kernel  $s_t$  on  $X$  is given by

$$s_t(\exp H) = (\mathcal{H}^{-1}w_t)(\exp H), \quad H \in \overline{\mathfrak{a}_+}$$

where

$$w_t(\lambda) = e^{it|\rho|^2} e^{it|\lambda|^2}, \quad \lambda \in \mathfrak{p}^*.$$

So, to obtain pointwise estimates of the kernel  $s_t$  which are crucial for our proofs, we need a manipulable expression of the inverse spherical Fourier transform  $\mathcal{H}^{-1}$ . This happens exactly in the two cases we deal with in the present work: the rank one case and the case  $G$  complex. The case of rank one symmetric spaces is treated in [6]. In Section 4 we treat the case  $G$  complex by exploiting the fact that the spherical Fourier transform boils down to the Euclidean Fourier transform on  $\mathfrak{p}$ . More precisely, the inverse spherical Fourier transform is given by the following formula:

$$(16) \quad \mathcal{H}^{-1}f(\exp H) = c\varphi_0(\exp H) \int_{\mathfrak{p}^*} f(\lambda) e^{i\lambda(H)} d\lambda, \quad H \in \mathfrak{p},$$

where  $\varphi_0$  is the basic spherical function [4, p. 1312].

### 3. AN ANALOGUE OF KUNZE-STEIN'S PHENOMENON ON LOCALLY SYMMETRIC SPACES

Let us recall that a central result in the theory of convolution operators on semisimple Lie groups is the Kunze-Stein phenomenon, which states that if  $p \in [1, 2)$ ,  $f \in L^2(G)$  and  $\kappa \in L^p(G)$ , then

$$(17) \quad \|f * \kappa\|_{L^2(G)} \leq C(p) \|f\|_{L^2(G)} \|\kappa\|_{L^p(G)},$$

(see [21, p. 3361]). This inequality was proved by Kunze and Stein [25] in the case when  $G = SL(2, \mathbb{R})$  and by Cowling [12] in the general case. In [19] Herz noticed that the inequality (17) can be sharpened if the kernel  $\kappa$  is  $K$ -bi-invariant. Indeed, the Herz's criterion [19] asserts that if  $p \geq 1$  and  $\kappa$  is a  $K$ -bi-invariant kernel, then

$$(18) \quad \begin{aligned} \| * |\kappa| \|_{L^p(G) \rightarrow L^p(G)} &\leq C \int_G |\kappa(g)| \varphi_{-i\rho_p}(g) dg \\ &= C \int_{\mathfrak{a}_+} |\kappa(\exp H)| \varphi_{-i\rho_p}(\exp H) \delta(H) dH, \end{aligned}$$

where  $\rho_p = |2/p - 1|\rho$ ,  $p \geq 1$ .

Note that for  $p = 2$ , the best we can obtain in the Euclidean case is the inequality

$$\| * |\kappa| \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq \|\kappa\|_{L^1(\mathbb{R}^n)},$$

while in the semisimple case we have that

$$(19) \quad \| * |\kappa| \|_{L^2(G) \rightarrow L^2(G)} \leq C \int_{\mathfrak{a}_+} |\kappa(\exp H)| \varphi_0(\exp H) \delta(H) dH.$$



Bearing in mind that

$$\|\kappa\|_{L^1(G)} = \int_{\mathfrak{a}_+} |\kappa(\exp H)| \delta(H) dH,$$

we deduce from the above norm estimates that for  $p = 2$ , the non-trivial gain over the Euclidean case is the factor  $\varphi_0(\exp H)$ .

Kunze-Stein's phenomenon is no more valid on locally symmetric spaces  $M = \Gamma \backslash G/K$ . In [27] Lohoué et Marias proved an analogue of this phenomenon for a class of locally symmetric spaces. More precisely, let  $\lambda_0$  be the bottom of the  $L^2$ -spectrum of  $-\Delta$  on  $M$ . We say that  $M$  possesses property (KS) if there exists a vector  $\eta_\Gamma$  on the Euclidean sphere  $S\left(0, (|\rho|^2 - \lambda_0)^{1/2}\right)$  of  $\mathfrak{a}^*$ , such that for all  $p \in (1, \infty)$ ,

$$(20) \quad \|\kappa\|_{L^p(M) \rightarrow L^p(M)} \leq \int_G |\kappa(g)| \varphi_{-i\eta_\Gamma}(g)^{s(p)} dg,$$

where

$$(21) \quad s(p) = 2 \min((1/p), (1/p')).$$

In [27] it is shown that  $M$  possesses property (KS) if it is contained in the following three classes:

(i).  $\Gamma$  is a lattice i.e.  $\text{vol}(\Gamma \backslash G) < \infty$ ,

(ii).  $G$  possesses Kazhdan's property (T). Recall that  $G$  has property (T) iff  $G$  has no simple factors locally isomorphic to  $SO(n, 1)$  or  $SU(n, 1)$ , [16, ch. 2]. In this case  $\Gamma \backslash G/K$  possesses property (KS) for all discrete subgroups  $\Gamma$  of  $G$ . Recall that  $H^n(\mathbb{H}) = Sp(n, 1)/Sp(n)$  and  $H^2(\mathbb{O}) = F_4^{-20}/Spin(9)$ . So,  $\Gamma \backslash H^n(\mathbb{H})$  and  $\Gamma \backslash H^2(\mathbb{O})$  have property (KS) for all discrete subgroups  $\Gamma$  of  $Sp(n, 1)$  and  $F_4^{-20}$  respectively.

Thus, from cases (i) and (ii) we deduce that all locally symmetric spaces  $\Gamma \backslash H^n(\mathbb{H})$  and  $\Gamma \backslash H^2(\mathbb{O})$  have property (KS).

On the contrary, the isometry groups  $SO(n, 1)$  and  $SU(n, 1)$  of real and complex hyperbolic spaces do not have property (T) and consequently the quotients  $\Gamma \backslash H^n(\mathbb{R})$  and  $\Gamma \backslash H^n(\mathbb{C})$  of infinite volume do not in general belong in the class (ii). The class (iii) below covers also this case.

(iii)  $\Gamma \backslash G$  is non-amenable. Note that since  $G$  is non-amenable, then  $\Gamma \backslash G$  is non-amenable if  $\Gamma$  is amenable. So, if  $\Gamma$  is amenable, then the quotients  $\Gamma \backslash H^n(\mathbb{R})$  and  $\Gamma \backslash H^n(\mathbb{C})$  possess property (KS) even if they have infinite volume. Note that if  $\Gamma$  is finitely generated and has subexponential growth, then  $\Gamma$  is amenable, [1, 15].

Let us now explain when a finitely generated group  $\Gamma$  has subexponential growth. Let  $A = \{a_1, a_2, \dots, a_m\}$  be a system of generators of

$\Gamma$ . The length  $|g|_A$  of  $g \in \Gamma$  with respect to  $A$  is the length  $n$  of the shortest representation of  $g$  in the form  $g = a_{i_1}^\pm a_{i_2}^\pm \cdots a_{i_n}^\pm$ ,  $a_{i_j} \in A$ . This depends on the set  $A$  but, for any two systems of generators  $A$  and  $B$ ,  $|g|_A$  and  $|g|_B$  are equivalent. The growth function of  $\Gamma$  with respect to the set  $A$  is the function

$$\gamma_\Gamma^A(n) = \# \{g \in G : |g|_A \leq n\},$$

where  $\#E$  denotes the cardinality of the set  $E$ . We say that  $\Gamma$  has subexponential growth if  $\gamma_\Gamma^A(n)$  grows more slowly than any exponential function.

#### 4. POINTWISE ESTIMATES OF THE SCHRÖDINGER KERNEL ON $X$

As it is already mentioned, the Schrödinger kernel  $s_t$  on  $X$  is given by

$$(22) \quad s_t(\exp H) = (\mathcal{H}^{-1} w_t)(\exp H), \quad H \in \overline{\mathfrak{a}_+},$$

where

$$w_t(\lambda) = e^{it|\rho|^2} e^{it|\lambda|^2}, \quad \lambda \in \mathfrak{p}^*.$$

Using (22) and the expression of the inverse spherical Fourier transform  $\mathcal{H}^{-1}$  in the case of rank one symmetric spaces, Anker, Pierfelice et Vallarino [6] obtained the following estimates of  $s_t$ :

$$(23) \quad |s_t(r)| \leq c \psi_1(t, r) e^{-|\rho|r},$$

where

$$(24) \quad \psi_1(t, r) = \begin{cases} |t|^{-n/2} (1+r)^{(n-1)/2}, & \text{if } |t| \leq 1+r, \\ |t|^{-3/2} (1+r), & \text{if } |t| > 1+r. \end{cases}$$

(See also [5, 28] for the case of the real hyperbolic space).

**Lemma 4.** *If  $G$  is complex, then the Schrödinger kernel  $s_t$  on  $X = G/K$  is given by*

$$(25) \quad s_t(\exp H) = c \varphi_0(\exp H) t^{-n/2} e^{it|\rho|^2} e^{-i|H|^2/4t}, \quad H \in \overline{\mathfrak{a}_+}, \quad t \in \mathbb{R},$$

where  $n = \dim \mathfrak{p} = \dim X$ .

*Proof.* As it is already mentioned, if  $G$  is complex, then the inverse spherical Fourier transform is given by

$$(26) \quad \mathcal{H}^{-1} f(\exp H) = c \varphi_0(\exp H) \int_{\mathfrak{p}^*} f(\lambda) e^{i\lambda(H)} d\lambda, \quad H \in \mathfrak{p},$$

where  $\varphi_0$  is the basic spherical function [4, p. 1312]. From (26) and (22) we get that

$$s_t(\exp H) = c\varphi_0(\exp H) \int_{\mathfrak{p}^*} e^{it(|\rho|^2+|\lambda|^2)} e^{i\lambda(H)} d\lambda, \quad H \in \mathfrak{p}.$$

Write

$$e^{it(|\rho|^2+|\lambda|^2)} e^{i\lambda(H)} = e^{it|\rho|^2} \prod_{j \leq n} e^{i\lambda_j^2} e^{i\lambda_j H_j},$$

and note that the function  $\lambda \rightarrow g(\lambda) = e^{i\lambda^2} e^{i\lambda H}$ ,  $\lambda \in \mathbb{C}$ , is analytic. So, we can compute the integral

$$I(t, H) := \int_{\mathbb{R}} e^{i\lambda^2} e^{i\lambda H} d\lambda$$

by changing the path of integration. In fact, we shall integrate on the first diagonal  $\gamma(\lambda) = e^{i\pi/4}\lambda$ ,  $\lambda \in \mathbb{R}$ , and get that

$$\begin{aligned} I(t, H) &= \int_{\mathbb{R}} e^{ite^{i(\pi/2)}\lambda^2} e^{ie^{i(\pi/4)}\lambda H} d\lambda = \int_{\mathbb{R}} e^{-t\lambda^2} e^{i\lambda(e^{i(\pi/4)}H)} d\lambda \\ &= (4\pi t)^{-1/2} e^{-(e^{i(\pi/4)}H)^2/4t} = (4\pi t)^{-1/2} e^{-iH^2/4t}, \end{aligned}$$

and (25) follows by bearing in mind that  $s_t$  is  $K$ -bi-invariant.  $\square$

**Proposition 5.** *If  $G$  is complex, then the Schrödinger kernel  $s_t$  on  $X = G/K$  satisfies the following estimates*

$$(27) \quad |s_t(\exp H)| \leq ct^{-n/2} (1 + |H|)^a e^{-\rho_m|H|}, \quad H \in \overline{\mathfrak{a}_+}, \quad t \in \mathbb{R},$$

for some constants  $c, a > 0$ .

*Proof.* Recall that for  $H \in \overline{\mathfrak{a}_+}$ ,

$$(28) \quad \varphi_0(\exp H) \leq c(1 + |H|)^a e^{-\rho(H)},$$

for some constants  $c, a > 0$ , [18]. Combining (28) with (25), we get

$$\begin{aligned} |s_t(\exp H)| &= c\varphi_0(\exp H) \leq c(1 + |H|)^a e^{-\rho(H)} \\ &\leq ct^{-n/2} (1 + |H|)^a e^{-\rho_m|H|}. \end{aligned}$$

$\square$

**Remark 6.** *As it is already mentioned, if  $X$  has rank one, then  $|\rho| = \rho_m$ . Also, in both cases,  $s_t$  is  $K$ -bi-invariant. So, the estimates (23) and (27) of  $s_t$  can be written in the following form:*

$$(29) \quad |s_t(\exp H)| \leq c\psi(t, H) e^{-\rho_m|H|}, \quad H \in \overline{\mathfrak{a}_+}, \quad t \in \mathbb{R},$$

where

$$(30) \quad \psi(t, H) = \begin{cases} \psi_1(t, H), & \text{if } M \text{ has rank one,} \\ \psi_2(t, H) := t^{-n/2} (1 + |H|)^a, & \text{if } G \text{ is complex,} \end{cases}$$

and  $\psi_1$  is defined in (24).

### 5. NORM ESTIMATES OF THE SCHRÖDINGER KERNEL ON $M$

Recall that the Schrödinger operator  $\widehat{S}_t$  on  $M$  is initially defined as a convolution operator

$$(31) \quad \widehat{S}_t f(x) = \int_G s_t(y^{-1}x) f(y) dy, \quad f \in C_0^\infty(M).$$

Set  $s_t(x, y) = s_t(y^{-1}x)$  and

$$(32) \quad \widehat{s}_t(x, y) = \sum_{\gamma \in \Gamma} s_t(x, \gamma y) = \sum_{\gamma \in \Gamma} s_t((\gamma y)^{-1}x).$$

**Proposition 7.** *For all groups  $\Gamma$  with  $\delta(\Gamma) < \rho_m$ , the series (32) converges and the Schrödinger operator  $\widehat{S}_t$  on  $M$  is given by*

$$(33) \quad \widehat{S}_t f(x) = \int_M \widehat{s}_t(x, y) f(y) dy.$$

*Proof.* Use the Cartan decomposition and write  $(\gamma y)^{-1}x = k_\gamma \exp H_\gamma k'_\gamma$ . Then, since  $s_t$  is  $K$ -bi-invariant,  $s_t((\gamma y)^{-1}x) = s_t(\exp H_\gamma)$  and the estimate (29) implies that

$$\begin{aligned} |\widehat{s}_t(x, y)| &\leq \sum_{\gamma \in \Gamma} |s_t((\gamma y)^{-1}x)| \leq \sum_{\gamma \in \Gamma} |s_t(\exp H_\gamma)| \\ &\leq c \sum_{\gamma \in \Gamma} \psi(t, |H_\gamma|) e^{-\rho_m |H_\gamma|}. \end{aligned}$$

If  $M$  has rank one, then by (30) and (24) we have that for every  $\varepsilon > 0$

$$\begin{aligned} |\widehat{s}_t(x, y)| &\leq \sum_{\gamma \in \Gamma} \psi(t, |H_\gamma|) e^{-\rho_m |H_\gamma|} \\ &\leq c|t|^{-n/2} \sum_{\{\gamma \in \Gamma: |t| \leq 1 + |H_\gamma|\}} e^{(\varepsilon - \rho_m)|H_\gamma|} + c|t|^{-3/2} \sum_{\{\gamma \in \Gamma: |t| > 1 + |H_\gamma|\}} e^{(\varepsilon - \rho_m)|H_\gamma|} \\ &\leq c|t|^{-n/2} \sum_{\gamma \in \Gamma} e^{(\varepsilon - \rho_m)d(0, (\gamma y)^{-1}x)} + c|t|^{-3/2} \sum_{\gamma \in \Gamma} e^{(\varepsilon - \rho_m)d(0, (\gamma y)^{-1}x)} \\ &\leq c(|t|^{-3/2} + |t|^{-n/2}) \sum_{\gamma \in \Gamma} e^{(\varepsilon - \rho_m)d(x, \gamma y)} \\ &\leq c(|t|^{-3/2} + |t|^{-n/2}) P_{\rho_m - \varepsilon}(x, y) < \infty \end{aligned}$$

provided that  $\rho_m > \delta(\Gamma)$ .

The proof of the case  $G$  complex is similar and then omitted.

Let us now prove (33). Since  $s_t$  and  $f$  are right  $K$ -invariant, from (31) we get that

$$\widehat{S}_t f(x) = \int_X s_t(x, y) f(y) dy.$$

Now, since  $f$  is left  $\Gamma$ -invariant, by Weyl's formula we find that

$$\begin{aligned} \widehat{S}_t(f)(x) &= \int_X f(y) s_t(x, y) dy = \int_{\Gamma \backslash X} \left( \sum_{\gamma \in \Gamma} f(\gamma y) s_t(x, \gamma y) \right) dy \\ &= \int_M f(y) \widehat{s}_t(x, y) dy. \end{aligned}$$

□

Next, we prove norm estimates for the Schrödinger kernel on  $M$  which are crucial for the proofs of our results.

**Proposition 8.** *If  $\delta(\Gamma) < \rho_m$ , then for any  $q > 2$  and  $x \in X$ ,*

$$\begin{aligned} \|\widehat{s}_t(x, \cdot)\|_{L^q(M)} &\leq c \|s_t(x, \cdot)\|_{L^q(X)} \\ (34) \quad &\leq c \Psi(t), \quad t \in \mathbb{R}, \end{aligned}$$

where

$$\Psi(t) = \begin{cases} |t|^{-n/2}, & \text{if } |t| \leq 1, \\ |t|^{-3/2}, & \text{if } |t| > 1, \end{cases}$$

in the rank one case, and

$$\Psi(t) = |t|^{-n/2}, \quad t \in \mathbb{R},$$

in the case when  $G$  is complex.

*Proof.* Fix  $N_0 \in \mathbb{N}$  and write  $\Gamma = \Gamma_1 \cup \Gamma_2$ , where

$$\Gamma_1 = \{\gamma \in \Gamma : d(x, \gamma y) > N_0\} \text{ and } \Gamma_2 = \{\gamma \in \Gamma : d(x, \gamma y) \leq N_0\}.$$

Since  $\Gamma$  is a discrete group, then  $\Gamma_2$  is a finite set. On the other hand, since the series  $\sum_{\gamma \in \Gamma} s_t(x, \gamma y)$  is convergent, we can choose  $N_0$  such that

$$\left| \sum_{\gamma \in \Gamma} s_t(x, \gamma y) \right| \leq 2 \left| \sum_{\gamma \in \Gamma_2} s_t(x, \gamma y) \right|.$$

But  $\Gamma_2$  is a finite set, so, for  $q > 2$ , we have that

$$\left| \sum_{\gamma \in \Gamma_2} s_t(x, \gamma y) \right|^q \leq c \sum_{\gamma \in \Gamma_2} |s_t(x, \gamma y)|^q.$$

It follows that

$$(35) \quad \left| \sum_{\gamma \in \Gamma} s_t(x, \gamma y) \right|^q \leq c \sum_{\gamma \in \Gamma_2} |s_t(x, \gamma y)|^q \leq c \sum_{\gamma \in \Gamma} |s_t(x, \gamma y)|^q.$$

Using (35) and Weyl's formula, we have that

$$\begin{aligned} \|\widehat{s}_t(x, \cdot)\|_{L^q(M)}^q &\leq 2 \int_M \left| \sum_{\gamma \in \Gamma_2} s_t(x, \gamma y) \right|^q dy \leq c \int_M \sum_{\gamma \in \Gamma} |s_t(x, \gamma y)|^q dy \\ &= c \int_X |s_t(x, y)|^q dy = c \|s_t(x, \cdot)\|_{L^q(X)}^q. \end{aligned}$$

Assume now that  $M$  has rank one. Using that  $s_t(x, y)$  is radial, from (13) we have that

$$(36) \quad \|s_t(x, \cdot)\|_{L^q(X)}^q \leq c \int_{\mathbb{R}^+} |s_t(r)|^q e^{2|\rho|r} dr.$$

Next, using the estimates (23) of  $s_t(x, y)$  and the fact  $q > 2$ , we shall prove (34). Indeed, choose  $\epsilon > 0$  such that  $\epsilon < (q - 2)|\rho|$ . Then if  $|t| \geq 1$ , by (36) we get

$$\begin{aligned} \|s_t(x, \cdot)\|_{L^q(X)}^q &= \int_X |s_t(x, y)|^q dy \\ &\leq c |t|^{-3q/2} \int_0^{|t|-1} e^{r(\epsilon - q|\rho| + 2|\rho|)} dr + c |t|^{-nq/2} \int_{|t|-1}^\infty e^{r(\epsilon - q|\rho| + 2|\rho|)} dr \\ &\leq c \left( |t|^{-3q/2} + |t|^{-nq/2} \right) \int_0^\infty e^{-r((q-2)|\rho| - \epsilon)} dr \\ &\leq c |t|^{-3q/2}. \end{aligned}$$

The case  $|t| \leq 1$ , as well as the case  $G$  complex, are similar and thus omitted.  $\square$

## 6. DISPERSIVE ESTIMATES FOR THE SCHRÖDINGER OPERATOR ON $M$ .

In this section we prove dispersive estimates for the Schrödinger operator  $\widehat{S}_t$  on locally symmetric spaces in the class  $(S_0)$ . To begin with, we make use of the estimates of the  $L^q$ -norm of the kernel  $\widehat{s}_t(x, y)$  obtained in the previous section, in order to estimate the operator norms of  $\widehat{S}_t$ .

**Lemma 9.** *If  $\delta(\Gamma) < \rho_m$ , then for any  $q > 2$*

$$(37) \quad \|\widehat{S}_t\|_{L^1(M) \rightarrow L^q(M)} \leq c \Psi(t), \quad t \in \mathbb{R},$$

and

$$(38) \quad \|\widehat{S}_t\|_{L^{q'}(M) \rightarrow L^\infty(M)} \leq c\Psi(t),$$

where  $\Psi(t)$  is defined in Proposition 8.

*Proof.* Assume that  $G$  is complex. If  $f \in L^1(M)$  and  $q > 2$ , then Minkowski's inequality and Proposition 8 imply that

$$\begin{aligned} \|\widehat{S}_t f\|_{L^q(M)} &= \left( \int_M \left| \int_M \widehat{s}_t(x, y) f(y) dy \right|^q dx \right)^{1/q} \\ &\leq \int_M \left( \int_M |\widehat{s}_t(x, y) f(y)|^q dx \right)^{1/q} dy \\ &= \int_M |f(y)| \left( \int_M \widehat{s}_t(x, y)^q dx \right)^{1/q} dy \\ &\leq \|f\|_{L^1(M)} \sup_x \|s_t(x, \cdot)\|_{L^q(X)} \\ &\leq c|t|^{-n/2} \|f\|_{L^1(M)}. \end{aligned}$$

The estimate (38) follows from (37) by duality. The rank one case is similar.  $\square$

Next, we make use of the analogue of Kunze-Stein phenomenon for locally symmetric spaces proved in [27] and presented in Section 2, in order to obtain the estimate of the norm  $\|\ast s_t\|_{L^{q'}(M) \rightarrow L^q(M)}$ . For that, as in [6, p. 988], we consider the spaces  $A_q = L^{q/2}(\overline{\mathfrak{a}}_+, \varphi_0 \delta)$ ,  $q \in [2, \infty)$ , of all  $K$ -bi-invariant functions  $f$  on  $G$ , that satisfy

$$\|f\|_{A_q}^{q/2} = \int_{\overline{\mathfrak{a}}_+} |f(\exp H)|^{q/2} \varphi_0(\exp H) \delta(H) dH < \infty.$$

For  $p = \infty$ , we take  $A_\infty = L^\infty(\overline{\mathfrak{a}}_+)$ .

The following analogue of Theorem 4.2 of [6] is a consequence of Kunze and Stein phenomenon.

**Theorem 10.** *Assume that  $M \in (S)$ . If  $\kappa \in S(K \backslash G / K)$ , then for all  $q \geq 2$ ,*

$$(39) \quad L^{q'}(M) \ast A_q \subset L^q(M).$$

*In other words, there exists a  $c_q > 0$  such that*

$$(40) \quad \|f \ast \kappa\|_{L^q(M)} \leq c_q \|\kappa\|_{A_q} \|f\|_{L^{q'}(M)}.$$

*Proof.* Since  $\kappa \in A_2$ , then from (20) it follows that

$$\begin{aligned} \| * \kappa \|_{L^2(M) \rightarrow L^2(M)} &\leq \int_{\overline{\mathfrak{a}_+}} |\kappa(\exp H)| |\varphi_0(\exp H) \delta(H) dH \\ (41) \qquad \qquad \qquad &= \|\kappa\|_{A_2}. \end{aligned}$$

Furthermore,  $\kappa \in A_\infty$ , so, for every  $f \in L^1(M)$  and  $x \in G$ , we have

$$|f * \kappa(x)| \leq \int_G |\kappa(\overline{g}^{-1}x)| |f(\overline{g})| dg \leq \|\kappa\|_\infty \|f\|_{L^1(M)},$$

i.e.

$$(42) \qquad \qquad \qquad \| * \kappa \|_{L^1(M) \rightarrow L^\infty(M)} \leq \|\kappa\|_{A_\infty}.$$

From (41) and (42), it follows that

$$L^2(M) * A_2 \subset L^2(M),$$

and

$$L^1(M) * A_\infty \subset L^\infty(M).$$

By interpolating between the case  $q = 2$  and  $q = \infty$ , we obtain that for any  $\theta \in (0, 1)$

$$(43) \qquad [L^2(M), L^1(M)]_\theta * [A_2, A_\infty]_\theta \subset [L^2(M), L^\infty(M)]_\theta.$$

Choose  $\theta = 2/q < 1$ . Then

$$(44) \qquad [L^2(M), L^1(M)]_\theta = L^{p_\theta}(M) = L^{q'}(M),$$

since  $\frac{1}{p_\theta} = \frac{\theta}{2} + \frac{1-\theta}{1} = 1 - \frac{\theta}{2} = \frac{1}{q'}$ .

Similarly,

$$(45) \qquad [A_2, A_\infty]_\theta = [L^1(\overline{\mathfrak{a}_+}, \varphi_0 \delta), L^\infty(\overline{\mathfrak{a}_+})]_\theta = L^{q/2}(\overline{\mathfrak{a}_+}, \varphi_0 \delta) = A_q,$$

and

$$(46) \qquad [L^2(M), L^\infty(M)]_\theta = L^{q\theta}(M) = L^q(M).$$

Putting together (43), (44) and (45) we get that

$$L^{q'}(M) * A_q \subset L^q(M).$$

□

**Lemma 11.** *Assume that  $M \in (S_0)$ . If  $q > 2$ , then*

$$\| * s_t \|_{L^{q'}(M) \rightarrow L^q(M)} \leq c(q) \Psi(t),$$

where  $\Psi$  is defined in Proposition 8.



*Proof.* Assume that  $M$  has rank one. Let  $\omega_R$ ,  $R \in \mathbb{R}$ , be an even  $C^\infty$  cut-off function on  $\mathbb{R}$ , such that  $\omega_R(r) = 1$ , for  $|r| \leq R$  and  $\omega_R(r) = 0$ , for  $|r| > 2R$ .

Set  $s_t^R(r) = \omega_R(r) s_t(r)$ . Then, from Theorem 10, it follows that for  $q > 2$  and any  $R > 0$ , we have that

$$\|*s_t^R\|_{L^{q'}(M) \rightarrow L^q(M)} \leq c \|s_t^R\|_{A_q} = c \left( \int_{\mathbb{R}_+} |s_t^R(r)|^{q/2} \varphi_0(r) \delta(r) dr \right)^{2/q}.$$

Using the estimates (23) of  $s_t(r)$ , and the estimates

$$\varphi_0(r) \leq c(1+r)^\alpha e^{-r|\rho|} \text{ and } \delta(r) \leq e^{2r|\rho|},$$

we get that for  $q > 2$  and  $|t| > 1$ ,

$$\|s_t^R\|_{A_q} < ct^{-3/2}, \text{ for any } R > 0.$$

Letting  $R \rightarrow \infty$ , we get that  $s_t \in A_q$  and that

$$\|s_t\|_{A_q} < ct^{-3/2}, \text{ for } |t| > 1.$$

The proof in the case  $G$  complex is similar and then omitted.  $\square$

**6.1. Proof of the results.** Once we have established the necessary ingredients for the proof of dispersive estimates of the Schrödinger operator  $\hat{S}_t$  on  $M \in (S)$ , mainly the norm estimates of the kernel  $\hat{s}_t(x, y)$  obtained in Proposition 8 and the estimates of the operator norms of  $\hat{S}_t$  obtained in Lemmata 9 and 11, we give the proofs of Theorems 2 and 3 and present their applications to the study of well-posedness and scattering for NLS equations.

Having obtained the proofs of the ingredients mentionned above, the proofs become standard and they are similar to the proofs of the corresponding results in [5, 6]. Consequently we shall be brief and present only their main lines or simply we will omit them.

*Proof of Theorem 2.* Assume that  $G$  is complex. Then, for  $|t| < 1$  and  $2 < q, \tilde{q} \leq \infty$ , using the norm estimates of the kernel  $\hat{s}_t(x, y)$  we proved in Lemma 9, we get that

$$\begin{aligned} \|\hat{S}_t\|_{L^1(M) \rightarrow L^q(M)} &\leq c|t|^{-n/2}, \\ \|\hat{S}_t\|_{L^{\tilde{q}'}(M) \rightarrow L^\infty(M)} &\leq c|t|^{-n/2}. \end{aligned}$$

Also, by the spectral theorem, we have that

$$\|\hat{S}_t\|_{L^2(M) \rightarrow L^2(M)} = 1.$$

By interpolation we get that there exists a constant  $c > 0$  such that

$$\|\hat{S}_t\|_{L^{\tilde{q}'}(M) \rightarrow L^q(M)} \leq c|t|^{-n \max\{\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\tilde{q}}\}}, \text{ if } |t| < 1.$$

Similarly, for  $|t| \geq 1$ , by Lemmata 9, we have that

$$\begin{aligned}\|\widehat{S}_t\|_{L^1(M) \rightarrow L^q(M)} &\leq c|t|^{-n/2}, \\ \|\widehat{S}_t\|_{L^{\bar{q}'}(M) \rightarrow L^\infty(M)} &\leq c|t|^{-n/2}.\end{aligned}$$

Further, using Kunze and Stein, we proved in Lemma 11 that

$$\|\widehat{S}_t\|_{L^{q'}(M) \rightarrow L^q(M)} \leq c\|s_t\|_{A_q} \leq c|t|^{-n/2}.$$

By interpolation it follows that for  $|t| \geq 1$ ,

$$\|\widehat{S}_t\|_{L^{\bar{q}'}(M) \rightarrow L^q(M)} \leq c|t|^{-n/2}.$$

The rank one case is similar and then omitted.  $\square$

*Proof of Theorem 3.* As it is already mentioned in the Introduction, to prove the Strichartz estimates

$$(47) \quad \|u\|_{L_t^p L_x^q} \leq c \left\{ \|f\|_{L_x^2} + \|F\|_{L_t^{\bar{p}'} L_x^{\bar{q}'}} \right\},$$

for the solutions  $u(t, x)$  of the Cauchy problem

$$(48) \quad \begin{cases} i\partial_t u(t, x) + \Delta u(t, x) = F(t, x), & t \in \mathbb{R}, \ x \in M, \\ u(0, x) = f(x), \end{cases}$$

we shall combine the dispersive estimates of the operator  $e^{it\Delta}$  obtained in Theorem 2 with the classical  $TT^*$  method. This method consists in proving  $L_t^{p'} L_x^{q'} \rightarrow L_t^p L_x^q$  boundedness of the operator

$$TT^*F(t, x) = \int_{-\infty}^{+\infty} \widehat{S}_{t-s}F(s, x)ds$$

and of its truncated version

$$\widetilde{TT}^*F(t, x) = \int_0^t \widehat{S}_{t-s}F(s, x)ds,$$

for all admissible indices  $(p, q)$ . For that we shall make also use of the fact that the solutions of (48) are given by Duhamel's formula:

$$(49) \quad u(t, x) = e^{it\Delta}f(x) - i \int_0^t e^{i(t-s)\Delta}F(s, x)ds.$$

Assume that the pairs  $(p, q)$  satisfy

$$\frac{1}{q} \in \left( \left( \frac{1}{2} - \frac{1}{n} \right), \frac{1}{2} \right) \text{ and } \frac{1}{p} \in \left( \left( \frac{1}{2} - \frac{1}{q} \right) \frac{n}{2}, \frac{1}{2} \right).$$

From (49) it follows that to finish the proof of the theorem, it is enough to show that

$$(50) \quad \left\| \int_{-\infty}^{+\infty} \widehat{S}_{t-s} F(s) ds \right\|_{L_t^p L_x^q} \leq c \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}$$

and

$$(51) \quad \left\| \widehat{S}_t f(s) \right\|_{L_t^p L_x^q} \leq c \|f\|_{L_x^2}.$$

We give only the proof of (50). The proof of (51) is similar. We have that

$$\begin{aligned} \left\| \int_{-\infty}^{+\infty} \widehat{S}_{t-s} F(s) ds \right\|_{L_x^q} &\leq \int_{-\infty}^{+\infty} \left\| \widehat{S}_{t-s} F(s) \right\|_{L_x^q} ds \\ &\leq \int_{-\infty}^{+\infty} \left\| \widehat{S}_{t-s} \right\|_{L_x^{q'} \rightarrow L_x^q} \|F(s)\|_{L_x^{q'}} ds. \end{aligned}$$

Assume that  $M$  has rank one. Then from Theorem 2 we get that

$$\begin{aligned} \left\| \int_{-\infty}^{+\infty} \widehat{S}_{t-s} F(s) ds \right\|_{L_t^p L_x^q} &\leq \left\| \int_{|t-s| \geq 1} |t-s|^{-3/2} \|F(s)\|_{L_x^{q'}} ds \right\|_{L_t^p} \\ &\quad + \left\| \int_{|t-s| \leq 1} |t-s|^{-(1/2-1/q)n} \|F(s)\|_{L_x^{q'}} ds \right\|_{L_t^p} \\ &=: I_1 + I_2. \end{aligned}$$

To estimate  $I_1$  and  $I_2$  we consider the operators

$$T_1(f)(t) = \int_{|t-s| \geq 1} |t-s|^{-3/2} f(s) ds$$

and

$$T_2(f)(t) = \int_{|t-s| \leq 1} |t-s|^{-(1/2-1/q)n} f(s) ds.$$

Note the kernel  $k_1(u) = |u|^{-3/2} \chi_{\{|u| \geq 1\}}$  of  $T_1$  is bounded on  $L^1$ . So,  $T_1$  is bounded from  $L^{p'}$  to  $L^p$  for every  $p \in [2, \infty]$ . Similarly,  $k_2(u) = |u|^{-(1/2-1/q)n} \chi_{\{|u| \leq 1\}}$  is bounded on  $L^1$  if  $\frac{1}{q} \in ((\frac{1}{2} - \frac{1}{n}), \frac{1}{2})$ . This implies that  $T_2$  is bounded from  $L^{p_1}$  to  $L^{p_2}$  for all  $p_1, p_2 \in (1, \infty)$ , such that  $0 \leq \frac{1}{p_1} - \frac{1}{p_2} \leq 1 - \left(\frac{1}{2} - \frac{1}{q}\right)n$ . So,

$$I_1, I_2 \leq c \|F(s)\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}} ,$$

for all admissible pairs  $(p, q)$  and  $(\tilde{p}, \tilde{q})$  as above. The proof of the case  $G$  complex is similar.  $\square$

**6.2. Applications of Strichartz estimates.** In this section we apply Strichartz estimates to study well-posedness and scattering for NLS equations.

Consider the Cauchy problem for the inhomogeneous Schrödinger equation on  $M$ :

$$(52) \quad \begin{cases} i\partial_t u(t, x) + \Delta u(t, x) = F(u(t, x)), & t \in \mathbb{R}, \ x \in M, \\ u(0, x) = f(x), \end{cases}$$

and assume that  $F$  has a power-like nonlinearity of order  $\gamma$ , i.e.

$$|F(u)| \leq c|u|^\gamma, \quad |F(u) - F(v)| \leq c(|u|^{\gamma-1} + |v|^{\gamma-1})|u - v|.$$

Recall that the NLS is globally well-posed in  $L^2(M)$  if, for any bounded subset  $B$  of  $L^2(M)$ , there exists a Banach space  $Y$  continuously embedded into  $C(\mathbb{R}; L^2(M))$ , such that for any  $f \in B$ , the NLS has a unique solution  $u \in Y$  with  $u(0, x) = f(x)$  and the map  $T : B \rightarrow Y$ ,  $T(f) = u$  is continuous. Here, as in [5], we take

$$Y = Y_\gamma = C(\mathbb{R}; L^2(M)) \cap L^{\gamma+1}(\mathbb{R}; L^{\gamma+1}(M)),$$

which is a Banach space for the norm

$$\|u\|_{Y_\gamma} = \|u\|_{L_t^\infty L_x^2} + \|u\|_{L_t^{\gamma+1} L_x^{\gamma+1}},$$

and  $B = B(0, \varepsilon) \subset L^2(M)$ .

We have the following result.

**Theorem 12.** *Assume that  $M \in (S_0)$  and that  $F$  has a power-like nonlinearity of order  $\gamma$ . If  $\gamma \in (1, 1 + \frac{4}{n}]$ , then the NLS (52) is globally well-posed for small  $L^2$  data.*

Also, one can apply Strichartz's estimates in order to show scattering for the NLS in the case of power-like nonlinearity of order  $\gamma$  and for small  $L^2$  data.

**Theorem 13.** *Assume that  $M \in (S_0)$ . Consider the Cauchy problem (52) and assume that  $F$  has a power-like nonlinearity of order  $\gamma \in (1, 1 + \frac{4}{n}]$ . Then, for every global solution  $u$  corresponding to small  $L^2$  data, there exists  $u_\pm \in L^2(M)$  such that*

$$\|u(t) - e^{it\Delta} u_\pm\|_{L^2(M)} \rightarrow 0, \quad \text{as } t \rightarrow \pm\infty.$$

The proof of Theorems 12 and 13 are similar to the proofs of the corresponding results in [5] and then omitted.

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